On Generalized Radix Representations (Analytic Number Theory and Surrounding Areas)

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On Generalized Radix Representations

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You can find details and more material about the here studied kind of questions in the following papers:

[1] S. Akiyama and A. Pethő, *On canonical number systems*, Theor. Comp. Sci., 270 (2002), 921–933.

[2] S. Akiyama, H. Brunotte and A. Pethő, *Cubic CNS polynomials, notes on a conjecture of W.J. Gilbert*, J. Math. Anal. and Appl., **281** (2003), 402–415.

[3] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner, *On a generalization of the radix representation - a survey*, Fields Institute Communications, to appear.

[4] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner, *Generalized radix representations and dynamical systems I*, Acta Math. Hungar., submitted

1. Radix representation and two generalizations

Positive base: Let $g \geq 2$ be an integer. Then every $n \in \mathbf{Z}$ can be represented in the form

$$n = \pm \sum_{j=0}^{\ell} n_i g^i, \quad 0 \leq n_i < g.$$

Negative base: V. Grünwald (1885): Let $g \leq -2$ be an integer. Then every $n \in \mathbf{Z}$ can be represented in the form

$$n = \sum_{j=0}^{\ell} n_i g^i, \quad 0 \leq n_i < g.$$

1.1. β representation

A. Rényi (1957): Let $\beta > 1$ be a real number and $\mathcal{A} = \{0, 1, \cdots, \lfloor\beta\rfloor\}$ be the set of digits. Then each $\gamma \in [0, \infty)$ can be represented by

$$\gamma = a_m \beta^m + a_{m-1}\beta^{m-1} \cdots \qquad (1)$$

with $a_i \in \mathcal{A}$. This β-representation is usually not unique.

Assuming however that

$$0 \leq \gamma - \sum_{i=n}^{m} a_i \beta^i < \beta^n \qquad (2)$$

hold for all $n \leq m$ the β-representation become unique. For $\gamma \in [0, 1)$ this greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor\beta\gamma\rfloor$$

This concept is the topics of a lot of research:

- Description of the representations of 1, Erdős, Joó, Horváth.
- Characterization of univoque numbers, i.e. those β for which 1 has a unique representation, Daróczy, Kátai, Komornik, Loretti, Allouche, Cosnard.
- Connections with fractals: Rauzy, Thurston, Akiyama.
- Characterization of those β which leads to finite or eventually finite representations, Bertrand, Schmidt, Frougny, Solomyak, Hollander.
- Connection with radix representations based on linear recursive sequences, Zeckendorf, Fraenkel, Grabner, Tichy, Pethő.

1.2. CNS representation

Number rings: Knuth, Kátai, J. Szabó, B. Kovács, Gilbert (1960-1981):
Let $\mathbf{Z_K}$ be the ring of integers of the algebraic number field $\mathbf{K}$.

$\{\alpha, \mathcal{N}\}$; $\alpha \in \mathbf{Z_K}$, $\mathcal{N} = \{0, \ldots, |Norm(\alpha)|-1\}$ is called a *canonical number system* if every $\nu \in \mathbf{Z_K}$ can be represented in the form

$$\nu = \sum_{j=0}^{\ell} n_i \alpha^i, \quad n_i \in \mathcal{N}.$$

1.3. CNS polynomials

Observation: If $\mathbf{Z_K}$ is monogenic then $\mathbf{Z_K} = \mathbf{Z}[\alpha]$ for some $\alpha \in \mathbf{Z_K}$. This means $\mathbf{Z_K} \equiv \mathbf{Z}[x]/P(x)\mathbf{Z}[x]$, where $P(x)$ is the minimal polynomial of α.
Moreover, $\{\alpha, \mathcal{N}\}$ is a CNS in $\mathbf{Z}_{\mathbf{Q}(\alpha)}$ means nothing else that every coset of $\mathbf{Z}[x]/P(x)\mathbf{Z}[x]$ has an element (a representative) such that its coefficients is bounded by $|p_0| - 1$.

A monic polynomial $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0$ is called CNS polynomial if every coset of $\mathbf{Z}[x]/P(x)\mathbf{Z}[x]$ has an element

$$a_0 + a_1 x + \cdots + a_k x^k \tag{3}$$

such that $0 \le a_i < |p_0|$.

2. Comparison of the properties of greedy expansions and of CNS-polynomials

Let β be the root of $B(X) = X^d - b_1X^{d-1} - \cdots - b_d \in \mathbf{Z}[X]$.
Let Fin(β) be the set of positive real numbers having finite greedy expansion with respect to β. We say that $\beta > 1$ has property (F) if

$$\text{Fin}(\beta) = \mathbf{Z}[1/\beta] \cap [0, \infty).$$

Property (F)	**CNS-polynomials**
β is a Pisot number: $\beta > 1$, but its conjugates are < 1	the absolute value of all zeroes of $P(X)$ are larger than 1.
If $b_1 \ge \cdots \ge b_d \ge 1$, Frougny and Solomyak (1992)	If $p_{d-1} \le \cdots \le p_0$, $p_0 \ge 2$, B. Kovács (1981)
Characterization results if	
$b_1 > \lvert b_2\rvert + \cdots + \lvert b_d\rvert$, $b_d \ne 0$, Hollander (1996)	$p_0 > \lvert p_1\rvert + \cdots + \lvert p_{d-1}\rvert$, Akiyama, Pethő, (2002), Scheicher, Thuswaldner

3. Shift Radix Systems

Let $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping
$\tau_\mathbf{r} : \mathbb{Z}^d \to \mathbb{Z}^d$: if $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ then let

$$\tau_\mathbf{r}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

where $\mathbf{ra} = r_1a_1 + \cdots + r_da_d$, i.e. the inner product of $\mathbf{r}$ and $\mathbf{a}$.

Let $\mathbf{r}$ be fixed. We will show: $\mathbf{r}$ gives rise to a Pisot number β with property (F) as well as to a CNS-polynomial P iff

$$\textbf{for all } \mathbf{a} \in \mathbb{Z}^d\ \exists\, k > 0 \textbf{ with } \tau_\mathbf{r}^k(\mathbf{a}) = 0. \quad (4)$$

If (4) holds, we will call $\tau_\mathbf{r}$ a *shift radix system* (*SRS* for short). Hence SRS is a common generalization of greedy expansions with property (F) and CNS-polynomials.

3.1. Relation between SRS and β-expansions

Two basic definitions. Let

$$\mathcal{D}_d^0 := \left\{\mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d\, \exists k > 0 : \tau_\mathbf{r}^k(\mathbf{a}) = 0\right\} \text{ and}$$
$$\mathcal{D}_d := \left\{\mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d : \{\tau_\mathbf{r}^k(\mathbf{a})\}_{k\geq 0} \text{ is ultimately periodic}\right\}.$$

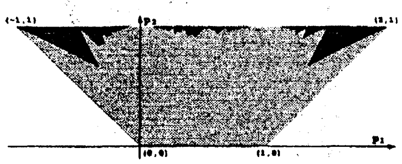
Now we can formulate the connection between SRS and greedy expansions.

Theorem 1 (Hollander,1996). *Let $\beta > 1$ be a Pisot number with minimal polynomial $X^d - b_1X^{d-1} - \cdots - b_{d-1}X - b_d$. Set*

$$r_1 := 1,$$
$$r_j := b_j\beta^{-1} + b_{j+1}\beta^{-2} + \cdots + b_d\beta^{j-d-1},$$

$(2 \leq j \leq d)$. Then β has property (F) if and only if $(r_d, \dots, r_2) \in \mathcal{D}_{d-1}^0$.

It is clear that $\mathcal{D}_1 = [-1, 1]$ and $\mathcal{D}_1^0 = [0, 1)$. To illustrate the difficulty of the characterization problem of $\mathcal{D}_d^0$ we show an approximation of $\mathcal{D}_2^0$.



An approximation of $\mathcal{D}_2^0$.

4. Relation between SRS and CNS-polynomials

This is a more delicate question.

Let $P(x) = p_dx^d + p_{d-1}x^{d-1} + \cdots + p_0 \in \mathbb{Z}[x]$ with $p_d = 1$. Every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element of form

$$A_0 + A_1x + \cdots + A_{d-1}x^{d-1}, \quad A_i \in \mathbb{Z}. \quad (5)$$

Let $\mathbb{Z}'[x] = \{A(x) \in \mathbb{Z}[x] : \deg A < d\}$ and

$$T(A) = \sum_{i=0}^{d-1} (A_{i+1} - qp_{i+1})x^i,$$

where $A_d = 0$ and $q = \lfloor A_0/p_0 \rfloor$.

Then $T : \mathbb{Z}'[x] \to \mathbb{Z}'[x]$ and

$$A = a_0 + xT(A), \text{ with } a_0 = A_0 - qp_0.$$

This backward division process can become:

- divergent $A(X) = -1$ for $P(X) = X^2 + 4X + 2$
$T^k_{X^2+4X+2}(-1) = -1, X + 4, -2X - 8, 4X + 16, \dots$ or
- ultimately periodic $A(X) = -1$ for $P(X) = X^2 - 2X + 2$
$T^k_{X^2-2X+2}(-1) = -1, X - 2, X - 1, X - 1, \dots$ or
- can terminate after finitely many steps $A(X) = -1$ for $P(X) = X^2 + 2X + 2$
$-1 = 1 + x^2 + x^3 + x^4.$

Let

$$\Pi(P) = \{A \ : \ T_P^\ell(A) = A \text{ for some } \ell > 0\}$$

denote the set of periodic points of the mapping T_P.
We always have $0 \in \Pi(P)$. With help of this set we define

$$\begin{aligned} \mathcal{C}_d^0 &= \{(p_0, p_1, \ldots, p_{d-1}) \in \mathbb{Z}^d : \\ &\quad \Pi(X^d + p_{d-1}X^{d-1} + \cdots + p_0) = \{0\}\} \quad \text{and} \\ \mathcal{C}_d &= \{(p_0, p_1, \ldots, p_{d-1}) \in \mathbb{Z}^d : \\ &\quad T_{X^d + p_{d-1}X^{d-1} + \cdots + p_0} \text{ has only finite orbits}\}. \end{aligned}$$

Clearly, we have $\mathcal{C}_d^0 \subset \mathcal{C}_d$.
The elements of $\mathcal{C}_d^0$ will be called CNS polynomials.

It is convenient to replace T_P by the conjugate mapping
$\tilde{T}_P \ : \ \mathbb{Z}^d \to \mathbb{Z}^d$ defined as

$$\tilde{T}_P(\mathbf{A}) = (A_1 - qp_1, \ldots, A_{d-1} - qp_{d-1}, -qp_d),$$

where $\mathbf{A} = (A_0, \ldots, A_{d-1})$ and $q = \lfloor A_0/p_0 \rfloor$.

4.1. Affect of a new representation

H. Brunotte (2000) and K. Scheicher and J. Thuswaldner (2001) observed that the basis transformation

$$\begin{aligned} \{1, x, \ldots, x^{d-1}\} &\to \{w_1, \ldots, w_d\}, \\ w_j &= \sum_{i=d-j+1}^{d} p_i x^{i+j-d-1} \end{aligned}$$

of R implies a nicer and much better applicable transformation than $\tilde{T}_P$ is. Indeed, if

$$\begin{aligned} A(x) &= \sum_{j=1}^{d} A_j w_j, \quad \text{then} \\ \tilde{T}_P(A) &= -tw_d + \sum_{j=1}^{d-1} A_{j+1} w_j, \text{ where} \\ t &= \left\lfloor \frac{p_1 A_d + \cdots + p_d A_1}{p_0} \right\rfloor. \end{aligned}$$

Hence, $\tilde{T}_P$ implies the mapping $\tau_P \ : \ \mathbb{Z}^d \to \mathbb{Z}^d$

$$\tau_P(\mathbf{A}) = \left(A_2, \ldots, A_d, -\left\lfloor \frac{p_1 A_d + \cdots + p_d A_1}{p_0} \right\rfloor\right)$$

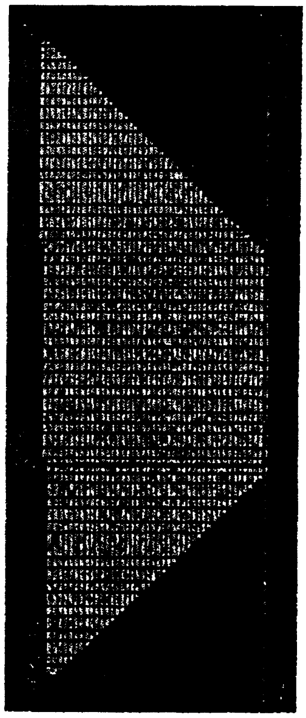
where $\mathbf{A} = (A_1, \ldots, A_d)$. The mapping τ_P will be called *Brunotte's mapping*.

Theorem 2. *Let $P(X) := X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$. Then $P(X)$ is a CNS polynomial (or belongs to $\mathcal{C}_d^0$) if and only if* $\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \ldots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^0$.

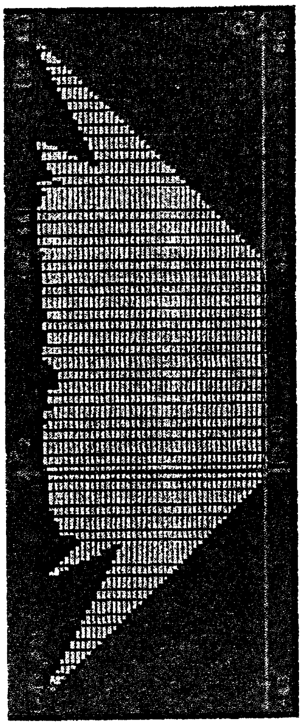
4.2. $\mathcal{C}_d^0$ for small d's

- $\mathcal{C}_1^0 = \{p_0 \ : \ p_0 \geq 2\}$, V. Grünwald
- $\mathcal{C}_2^0 = \{(p_0, p_1) \ : \ -1 \leq p_1 \leq p_0, p_0 \geq 2\}$, Kátai, Szabó, B. Kovács, Gilbert.
- Conjecture of Gilbert, 1981: $(p_0, p_1, p_2) \in \mathcal{C}_3^0$ if and only if

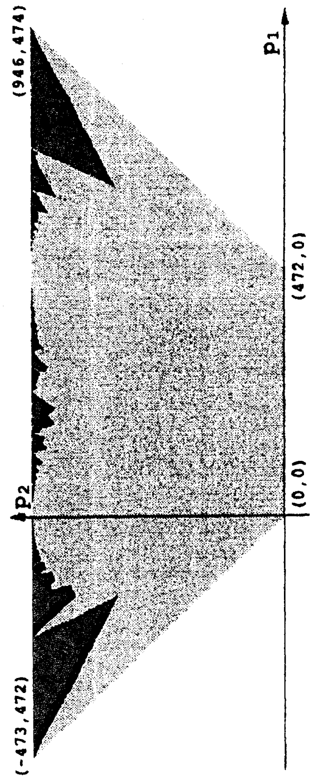
(*i*) $p_0 \geq 2$,
(*ii*) $p_2 \geq 0$,
(*iii*) $p_1 + p_2 \geq -1$,
(*iv*) $p_1 - p_2 \leq p_0 - 2$,
(*v*) $p_2 \leq \begin{cases} p_0 - 2, & \text{if} \quad p_1 \leq 0, \\ p_0 - 1, & \text{if} \quad 1 \leq p_1 \leq p_0 - 1, \\ p_0, & \text{if} \quad p_1 \geq p_0. \end{cases}$



Visualization of Gilbert's conjecture, $p_0 = 44$.



C_3^0 for $p_0 = 44$.



C_3^0 for $p_0 = 474$.

5. Basic properties of SRS

For a matrix M denote the spectral norm by $||M||$. For a vector v, $||v||$ shall denote the Euclidean norm.

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{D}_d$ let

$$R := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}. \quad (6)$$

Lemma 1. *Let $d \in \mathbf{N}$. If $\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{D}_d$ then the spectral radius of R is less than or equal to 1.*

In the opposite direction we get the following result.

Lemma 2. *Let $\mathbf{r} \in \mathbf{R}^d$ such that the spectral radius ρ of the matrix R given above is less than 1. Then $\mathbf{r} \in \mathcal{D}_d$.*

It is not to hard to prove the following statement

Theorem 3. *The sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ are Lebesgue measurable.*

5.1. Convexity property of $\tau_{\mathbf{r}}$

Theorem 4. *Let $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathbf{R}^d$ and $\mathbf{a} \in \mathbf{Z}^d$ be such that $\tau_{\mathbf{r}_1}(\mathbf{a}) = \cdots = \tau_{\mathbf{r}_k}(\mathbf{a})$. Let $\mathbf{s}$ be any convex linear combination of $r_1, \dots, r_k$. Then we have $\tau_{\mathbf{s}}(\mathbf{a}) = \tau_{\mathbf{r}_1}(\mathbf{a}) = \cdots = \tau_{\mathbf{r}_k}(\mathbf{a})$.*

Corollary 1. *Let $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathbf{R}$ have the same period, i.e. $\tau_{\mathbf{r}_1}^{\ell}(\mathbf{a}) = \cdots = \tau_{\mathbf{r}_k}^{\ell}(\mathbf{a}), \ell = 0, \dots, q$ and $\mathbf{a} = \tau_{\mathbf{r}_1}^{q}(\mathbf{a})$. Then if $\mathbf{s}$ is lying in the convex hull of $\mathbf{r}_1, \dots, \mathbf{r}_k$ the mapping $\tau_{\mathbf{s}}$ is periodic and has the same period as $\tau_{\mathbf{r}_1}$.*

For example, it is easy to check that for the plane vectors $r_1 = \left(\frac{381}{254}, \frac{253}{254}\right), r_2 = \left(\frac{421}{254}, \frac{253}{254}\right)$ and $r_3 = \left(\frac{344}{254}, \frac{176}{254}\right)$ the corresponding mappings have the same period $(-2, 1); 3, -2, 1, 1, -2$, hence the corresponding mapping for any point lying in the rectangle r_1, r_2, r_3 have this period.

5.2. Brunotte's algorithm

To decide $\mathbf{r} \in \mathcal{C}_d^0$ Brunotte gave an algorithm, which was realized independently by Scheicher and Thuswaldner. We give here a generalization for $\mathcal{D}_d^0$.

Theorem 5. *Suppose that there exists a set* $E \subset \mathbf{Z}^d$ *satisfying*
(i) *E contains* $2d$ *elements of the form* $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$.
(ii) $\tau_{\mathbf{r}}(E) \cup \tau_{\mathbf{r}}^*(E) \subset E$, *where* $\tau_{\mathbf{r}}^*(\mathbf{x}) = -\tau_{\mathbf{r}}(-\mathbf{x})$.
(iii) *For each* $\mathbf{a} \in E$ *there is some* $k > 0$ *such that* $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$.
Then $\mathbf{r} \in \mathcal{D}_d^0$.

An example: Let $\mathbf{r} = \left(\frac{2}{5}, -\frac{1}{5}\right)$.
Starting from $E_0 = \{(\pm 1, 0), (0, \pm 1)\}$ and using that
$\tau_{\mathbf{r}}(1,0) = (0,0), \tau_{\mathbf{r}}(-1,0) = (0,1), \tau_{\mathbf{r}}(0,1) = (1,1)$,
$\tau_{\mathbf{r}}(0,-1) = (-1,0)$ we get
$E_1 = \tau_{\mathbf{r}}(E_0) \cup \tau_{\mathbf{r}}^*(E_0) = E_0 \cup \{(0,0), (1,1), (-1,-1)\}$.
Now
$\tau_{\mathbf{r}}(1,1) = (1,0), \tau_{\mathbf{r}}(-1,-1) = (-1,1)$, hence we may take
$E_2 = \tau_{\mathbf{r}}(E_1) \cup \tau_{\mathbf{r}}^*(E_1) = E_1 \cup \{(1,-1), (-1,1)\}$.
Finally because
$\tau_{\mathbf{r}}(-1,1) = (1,1), \tau_{\mathbf{r}}(1,-1) = (-1,0)$, we get that
$E = E_2$ proves $\mathbf{r} \in \mathcal{D}_2^0$.

6. Lifting theorem

Let $d \in \mathbf{N}$ and

$$(a_{1+j}, \ldots, a_{d+j}) \in \mathbf{Z}^d, \qquad (0 \le j \le L-1), \quad (7)$$

with $a_{L+1} = a_1, \ldots, a_{L+d} = a_d$.
For which $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbf{R}^d$ these vectors form a period π of $\mathcal{D}_d$? By the definition of $\tau_{\mathbf{r}}$ this is the case if and only if the inequalities

$$0 \le r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (8)$$

hold simultaneously for all $0 \le j \le L-1$. They define a (possibly degenerated) polyhedron, which will be denoted by $\mathcal{P}(\pi)$.

Since $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $\tau_{\mathbf{r}}$ has 0 as its only period we conclude that

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \ne 0} \mathcal{P}(\pi)$$

where the union is extended over all families of vectors π of the shape (7). We call the family of (non-empty) polyhedra corresponding to this choice the *family of cutout polyhedra of* $\mathcal{D}_d^0$.

Let π be a period of $\mathcal{C}_d$ or $\mathcal{D}_d$ which corresponds to a non-degenerate cutout polyhedron. Then we call π a *non-degenerate period*. We will show that we can "lift" a non-degenerate period to higher dimensions.

Definition 1. *Let*

$$\pi : (a_1, \ldots, a_d); a_{d+1}, \ldots, a_L \qquad (9)$$

be a non-degenerate period of length L *of* $\mathcal{C}_d$ *or* $\mathcal{D}_d$. *Then we call*

$$l(\pi) : (a_1, a_2, \ldots, a_{d+1}); a_{d+2}, \ldots, a_L \qquad (10)$$

the lift of π *to* $d+1$.

Note that π and $l(\pi)$ have the same period length L.

Theorem 6 (Lifting Theorem). *Let $d \geq 1$ be an integer.*

(i) Let $p_0 \geq 2$ and let π be a non-degenerate period for $\mathcal{C}_d$. Then π is also a non-degenerate period for $\mathcal{D}_d$. More precisely, there exist $p_1, \dots, p_{d-1} \in \mathbf{Z}$ such that $(p_0, \dots, p_{d-1}) \in \operatorname{int}(\mathcal{P}'(\pi))$ and

$$\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}\right) \in \operatorname{int}(\mathcal{P}(\pi)).$$

(ii) Let π be a non-degenerate period of $\mathcal{D}_d$. Then its lift $\lambda := l(\pi)$ is a non-degenerate period of $\mathcal{C}_{d+1}$ for each sufficiently large p_0. More precisely, for all $(r_1, \dots, r_d) \in \operatorname{int}(\mathcal{P}(\pi))$ there exists $\varepsilon > 0$ such that for all $(p_0, \dots, p_d) \in \mathbf{Z}^{d+1}$ with

$$\max_{1 \leq k \leq d} \left| \frac{p_{d+1-k}}{p_0} - r_k \right| < \varepsilon$$

we have $(p_0, \dots, p_d) \in \operatorname{int}(\mathcal{P}'(\lambda))$.

7. Long periods

Consider the following family of edges.

$$\begin{array}{lrcll}
\alpha_k: & (-k-1, -n+k) & \to & (-n+k, k+1) & (0 \leq k \leq n-1), \\
\beta_k: & (-n+k, k+1) & \to & (k+1, n+1-k) & (0 \leq k \leq n-1), \\
\gamma_0: & (1, n+1) & \to & (n+1, 1), & \\
\gamma_k: & (k+1, n+1-k) & \to & (n+1-k, -k) & (1 \leq k \leq n-1), \\
\gamma_n: & (n+1, 1) & \to & (1, -n), & \\
\delta_k: & (n+1-k, -k) & \to & (-k, -n-1+k) & (1 \leq k \leq n-1).
\end{array}$$

With these edges we form the cycle

$$\zeta_n : \alpha_0\beta_0\gamma_0\gamma_n\alpha_{n-1}\beta_{n-1}\gamma_{n-1}\delta_{n-1}\alpha_{n-2} \cdots \alpha_1\beta_1\gamma_1\delta_1.$$

Note that δ_1 ends up in $(-1, -n)$. In this node α_0 starts. Thus ζ_n is indeed a cycle. We wonder whether there exists $\mathbf{r} := (x_n, y_n) \in \mathcal{D}_2$ such that $\tau_{\mathbf{r}}$ has ζ_n as a non-degenerate period. This is done in the following result.

Theorem 7. *Fix $n \in \mathbf{N}$, $n > 3$, and set $\mathbf{r} = (x_n, y_n) \in \mathbf{R}^2$ with*

$$x_n := 1 - \frac{1}{2n^2} + z_n \quad \text{and} \quad y_n := -\frac{2n+1}{2n(n+1)} + u_n,$$

where $|z_n|, |u_n| < 1/n^4$. Then ζ_n is a non-degenerate period of $\tau_{\mathbf{r}}$.

Since we can select n arbitrarily large and the length of the period ζ_n is $4n+1$ the previous theorem implies that there exist non-degenerate periods of arbitrarily large length for $\mathcal{D}_2$.

By a direct application of the Lifting Theorem we obtain.

Theorem 8. *Let $d \geq 2$ be an integer, fix $n \in \mathbf{N}$, $n > 3$. Then there exist some $\mathbf{r} \in \mathbf{R}^d$ such that $l^{d-2}(\zeta_n)$ is a non-degenerate period of $\tau_{\mathbf{r}}$. Since we can select n arbitrarily large and the length of the period $l^{d-2}(\zeta_n)$ is $4n+1$ this implies that there exist non-degenerate periods of arbitrarily large length of $\mathcal{D}_d$ and $\mathcal{C}_{d+1}$.*

Corollary 2. *Fix $n \in \mathbf{N}$, $n > 3$, and set $d \geq 2$ and $\mathbf{r} = (0, \dots, 0, x_n, y_n) \in \mathbf{R}^d$ with x_n, y_n as in Theorem 7. Then $l^{d-2}(\zeta_n)$ is a period of $\tau_{\mathbf{r}}$.*

8. Critical points

Definition 2. *Let* $\mathrm{x} \in \mathcal{D}_d$.

- *If there exists an open neighborhood of* x *which contains only finitely many cutout polyhedra then we call* x *a* **regular point**.
- *If each open neighborhood of* x *has nonempty intersection with infinitely many cutout polyhedra then we call* x *a* **weak critical point** *for* $\mathcal{D}_d$.
- *If for each open neighborhood* U *of* x *the set* $U \setminus \mathcal{D}_d^0$ *can not be covered by finitely many cutout polyhedra then* x *is called a* **critical point**.

We will show the existence of critical points for each $d \geq 2$. This shows that there is no way to characterize either of the sets $\mathcal{D}_d^0$ by finitely many cutouts if $d \geq 2$.

Lemma 3. *Let* x *be a weak critical point for* $\mathcal{D}_d$. *Then* $\mathrm{x} \in \partial\mathcal{D}_d$.

Lemma 4. *Let* $\{x_n\}_{n\geq 1}$ *and* $\{y_n\}_{n\geq 1}$ *be sequences with* $x_n < 1$, $y_n < 0$, $\lim x_n = 1$, $\lim y_n = 0$ *and* $1 - x_n = o(y_n)$. *Let* $\{a_m\}_{m\geq 1}$ *be a sequence of integers such that* $|a_i| < K$ *for some constant* K. *Then there exists* $N \in \mathbb{N}$ *such that*

$$0 \leq a_{i-1}x_n + a_i y_n + a_{i+1} < 1 \qquad (11)$$

can not hold for all i *if* $n \geq N$ *unless* $a_i = 0$ *for all* i *large enough. Thus nonzero periods whose elements are bounded by* K *can not occur in* $\mathcal{D}_d$ *for* $\tau_{(0,\ldots,0,x_n,y_n)}$ *if* n *is large enough.*

Theorem 9. *Let* $d \geq 2$. *Then* $\mathrm{K}_d := (0,\ldots,1,0) \in \mathbb{R}^d$ *is a critical point of* $\mathcal{D}_d$.

Problem 1. *Characterize the critical points of* $\mathcal{D}_d$. *Can one show that for a given* d *there exist only finitely many critical points?*